Self-organization of solitons for the dissipative Korteweg–de Vries equation

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The process by which self-organization occurs for solitons described by the Korteweg–de Vries equation with a viscous dissipation term is reinvestigated theoretically with the use of numerical simulations in a periodic system. It is shown that, during nonlinear interactions, two basic processes for the self-organization of solitons are energy transfer and selective dissipation among the eigenmodes of the dissipative operator. It is also clarified that an important process during nonlinear self-organization is an interchange between the dominant operators, which has hitherto been overlooked in conventional self-organization theories and which leads to a final self-similar coherent structure, determined uniquely by the dissipative operator.

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Theories have been described for self-organization in three-dimensional magnetohydrodynamic (MHD) plasmas [1, 2], two-dimensional MHD plasmas [3, 4], two-dimensional incompressible viscous fluids [5, 6], and solitons described by the Korteweg–de Vries (KdV) equation [7, 8]. As Ref. [8] has pointed out, these theories on self-organization all involve a logic that has in common the following four conceptual elements: (a) the system is described by dissipative nonlinear partial differential equations, (b) in the absence of dissipation, the system has three or more quadratic or higher-order conserved quantities, (c) when dissipation is introduced, one conserved quantity $A(q)$ decays faster than the others $B(q)$, where A and B are functionals of the field variables $q(t, \mathbf{x})$, this feature being known as "selective dissipation" between the invariants $A(q)$ and $B(q)$, and (d) the self-organized state is determined by minimizing A under the constraint that B is held constant.

Dynamical systems of interest having n variables $q_i(t, \mathbf{x})$, with $i = 1, 2, ..., n$, can generally be described by the following equations of motion:

$$\frac{\partial q_i}{\partial t} = L_i^N[\mathbf{q}] + L_i^D[\mathbf{q}] , \tag{1}$$

where $L_i^N[\mathbf{q}]$ and $L_i^D[\mathbf{q}]$ denote the nondissipative and dissipative dynamic operators, respectively, which may be either linear or nonlinear [9, 10]. In the conventional theories of self-organization, the nondissipative operators $L_i^N[\mathbf{q}]$ are assumed to be dominant throughout the entire self-organization process, with the dissipative operators $L_i^D[\mathbf{q}]$ assumed to be minor and thus capable of being handled perturbatively. Due to this implicit assumption, the self-organized states derived by the conventional theories have no dependence on the dissipative operators $L_i^D[\mathbf{q}]$ or on the dissipation parameters contained therein [8]. In this respect, the conventional self-organization theories are perturbative treatments. We note, however, that from the conceptual element (d), they lead to self-organized states that usually satisfy equilibrium equations of the form $L_i^N[\mathbf{q}] = 0$ [1–8]. By comparing this equilibrium equation with Eq. (1), we find that the nondissipative operators $L_i^N[\mathbf{q}]$ have little effect, compared with the dissipative operators $L_i^D[\mathbf{q}]$, at the phase when the self-organized states arise, since the time evolution of the dynamical system is determined dominantly by $L_i^D[\mathbf{q}]$. Hence there occurs an interchange between the dominant operators in the later phase of the self-organization process, with the spatial profiles of the final self-organized state being determined by the dissipative operators $L_i^D[\mathbf{q}]$. This analysis [11, 12] would suggest that, with respect to the interchange of dominant operators, there is a discrepancy with the logic of conventional theories and their four conceptual elements. The profile of the self-organized state will be one from among the set of equilibrium states that satisfy $L_i^N[q] = 0$. When the value of $L_i^N[q]$ is large, then the profile of q is significantly changing in time. When the system comes close to the equilibrium state, i.e., when the value of $L_i^N[q]$ is very small, then the dominant operator changes from $L_i^N[q]$ to $L_i^D[q]$. The final self-organized profile, which is one from among the set of equilibria satisfying $L_i^N[q] = 0$, will be determined uniquely by the operators $L_i^D[q]$, which are relatively dominant during the later phase of the self-organization process.

On the other hand, if we start from a definition for the self-organized state as that state for which the rate of change for the autocorrelations of instantaneous values is minimum, then the self-organized state so derived does depend explicitly on the dissipative operator of the dynamical system [9, 10, 13]. Some simulations [13, 14] have reported data that show the dependence of the self-organized state on the profile of the dissipation parameters. Those results also suggest that the dominant operator changes during the nonlinear self-organization process from the nondissipative nonlinear operators $L_i^N[\mathbf{q}]$ to the dissipative operators $L_i^D[\mathbf{q}]$.

Here we present a reinvestigation of the process by

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which self-organization occurs for solitons described by the KdV equation with a viscous dissipation term. Numerical simulations in a periodic system are employed. We will show that two basic processes for the self-organization of the solitons are energy transfer and selective dissipation among the eigenmodes of the dissipative operators $L_i^D[\mathbf{q}]$ during the nonlinear interactions. It will also be clarified that an important process during nonlinear self-organization is that of interchange between the dominant operators, which has hitherto not been recognized in conventional self-organization theories and which leads to the final self-similar coherent structure determined by the dissipative operator alone.

We investigate the self-organization process for solitons described by the following KdV equation with a viscous dissipation term:

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial x} + \delta^2\frac{\partial^3 q}{\partial x^3} = \eta\frac{\partial^2 q}{\partial x^2} \ . \tag{2}$$

Here δ is a constant, η is the coefficient of viscosity, and the nondissipative and dissipative operators $L_i^N[\mathbf{q}]$ and $L_i^D[\mathbf{q}]$ of Eq. (1) correspond, respectively, to the $-q\partial q/\partial x - \delta^2\partial^3 q/\partial x^3$ term and the $\eta\partial^2 q/\partial x^2$ term in Eq. (2). In the absence of dissipation ($\eta = 0$), it is known that the energy corresponding to the autocorrelation $W_{ii} = \int_0^b q(t,x)\cdot q(t,x)dx$ is conserved and each soliton behaves like a particle during nonlinear interactions, where b is the periodicity length. Since the rate of energy dissipation $\partial W_{ii}/\partial t$ due to the viscous term in Eq. (2) is $-2\int_0^b \eta(\partial q/\partial x)^2 dx$, the self-organized state q^*, which we will define here as that state for which the rate of change is minimum for the autocorrelation of instantaneous values [10], can be derived from the condition $\delta F = 0$, where F is a functional defined by $F \equiv -\partial W_{ii}/\partial t - \alpha W_{ii} = \int_0^b [2\eta(\partial q/\partial x)^2 - \alpha q^2]dx$, with α a Lagrange multiplier. Integrating by parts, we obtain

$$\delta F = -2\int_0^b \delta q\left[2\eta\frac{\partial^2 q}{\partial x^2} + \alpha q\right]dx = 0 \ , \tag{3}$$

where the periodicity constraint has been applied. We then obtain the Euler-Lagrange equation for an arbitrary variation δq, as follows:

$$\frac{\partial^2 q^*}{\partial x^2} + \lambda^2 q^* = 0 \ . \tag{4}$$

Here the parameter λ is defined by $\lambda^2 \equiv \alpha/2\eta$, and q^* denotes the self-organized state corresponding to a minimal rate of change of the autocorrelation. Using Eq. (4), we obtain the following equations:

$$\frac{\partial W ii^*}{\partial t} = -\alpha\, W_{ii}^* \ , \tag{5}$$

$$q^* = q_R^* e^{-(\alpha/2)t} \ , \tag{6}$$

$$q_R^* = A\sin(\lambda_1 x + \phi) \ , \tag{7}$$

Here $W_{ii}^* \equiv \int_0^b q^*(t,x)\cdot q^*(t,x)dx$; q_R^* is the solution of Eq. (4) for the self-organized state q^* under the periodicity condition; and the Lagrange multiplier is $\alpha = 2\eta\lambda_1^2$, with λ_1 the smallest positive eigenvalue that yields minimized rate of change for the autocorrelation of instantaneous values [10]. We expect theoretically from Eq. (6) that the time-decay constant of the profile q^* in the self-organized state is $\alpha/2$.

For the numerical simulations presented here, we used a new type of numerical scheme for hyperbolic equations, named the one-dimensional (1D) second KOND-H scheme, which has high numerical accuracy and stability through the use of the Kernel optimum nearly analytical discretization (KOND) algorithm [11, 12]. Double precision was employed for these calculations. Using the same process as shown in Fig. 10 of Ref. [12], we first obtained a numerical solution having four solitons per periodicity length for the KdV equation without the dissipative term, i.e., $\eta = 0$ in Eq. (2). Using this multisoliton solution as the initial profile, we investigated the self-organization process of the solitons in the presence of dissipation. A typical case with a periodicity length of $b = 50$, $\delta = 0.42$, and viscosity $\eta = 0.04$ was examined. In this case, since the smallest eigenvalue is $\lambda_1 = 2\pi/50$, the theoretical decay constant $\alpha/2$ in Eq. (6) is 0.632×10^{-3}, which was compared with the simulation results.

Figures 1(a)–(f) show the typical time evolution of the self-organization process for solitons with viscous dissipation, where the vertical scale is varied to accommodate the magnitude of the numerical amplitudes at each time. Figure 1(a) is the initial profile at $t = 0$ of four solitons per periodicity length, where the four solitons are labeled as q_{m1}, q_{m2}, q_{m3}, and q_{m4} in order from the largest to the smallest. When we check the numerical values for the nonlinear term $q\partial q/\partial x$ and the viscous term $\eta\partial^2 q/\partial x^2$ in Eq. (2) for this initial profile, the data show that $q\partial q/\partial x \simeq 0.14\times10^{-1} \gg \eta\partial^2 q/\partial x^2 \simeq 0.12\times10^{-2}$ at the point $x = 24.5$ where the viscous term is largest. In Fig. 1(b) at $t = 40$, interaction betweeen the first soliton q_{m1} and the fourth one q_{m4} is taking place, with the smaller soliton q_{m4} becoming absorbed into the larger one q_{m1}. In other words, the energy of the smaller soliton is transferred into the larger one during the interaction of two solitons when viscous dissipation is present. In Fig. 1(c) at $t = 220$, the third soliton q_{m3} is interacting with the first one q_{m1}, and energy transfer from the smaller soliton to the larger one is again occurring. At the same time, since the viscous dissipation tends to suppress the amplitudes of solitons and hence to broaden their widths, Fig. 1(c) also shows the second soliton q_{m2} beginning to interact with the first one q_{m1}. In Fig. 1(d) at $t = 900$, after the two smaller solitons q_{m3} and q_{m4} have been absorbed into the largest one q_{m1}, interaction and absorption of the second soliton q_{m2} into the first one q_{m1} continues to occur. The numerical data for Fig. 1(d) can be used to find that $q\partial q/\partial x \simeq 0.22\times10^{-4} \leq \eta\partial^2 q/\partial x^2 \simeq 0.26\times10^{-4}$ at the point $x = 22.5$ where the viscous term has its largest value. This result indicates that an interchange of the dominant operator between the nonlinear term $q\partial q/\partial x$ and the dissipative term $\eta\partial^2 q/\partial x^2$ will occur in this dynamical system at around the time $t = 900$ corresponding to Fig. 1(d). In Fig. 1(e) at $t = 1500$, all three smaller solitons, q_{m4}, q_{m3}, and q_{m2}, have been absorbed into the first one q_{m1} meaning that the energy

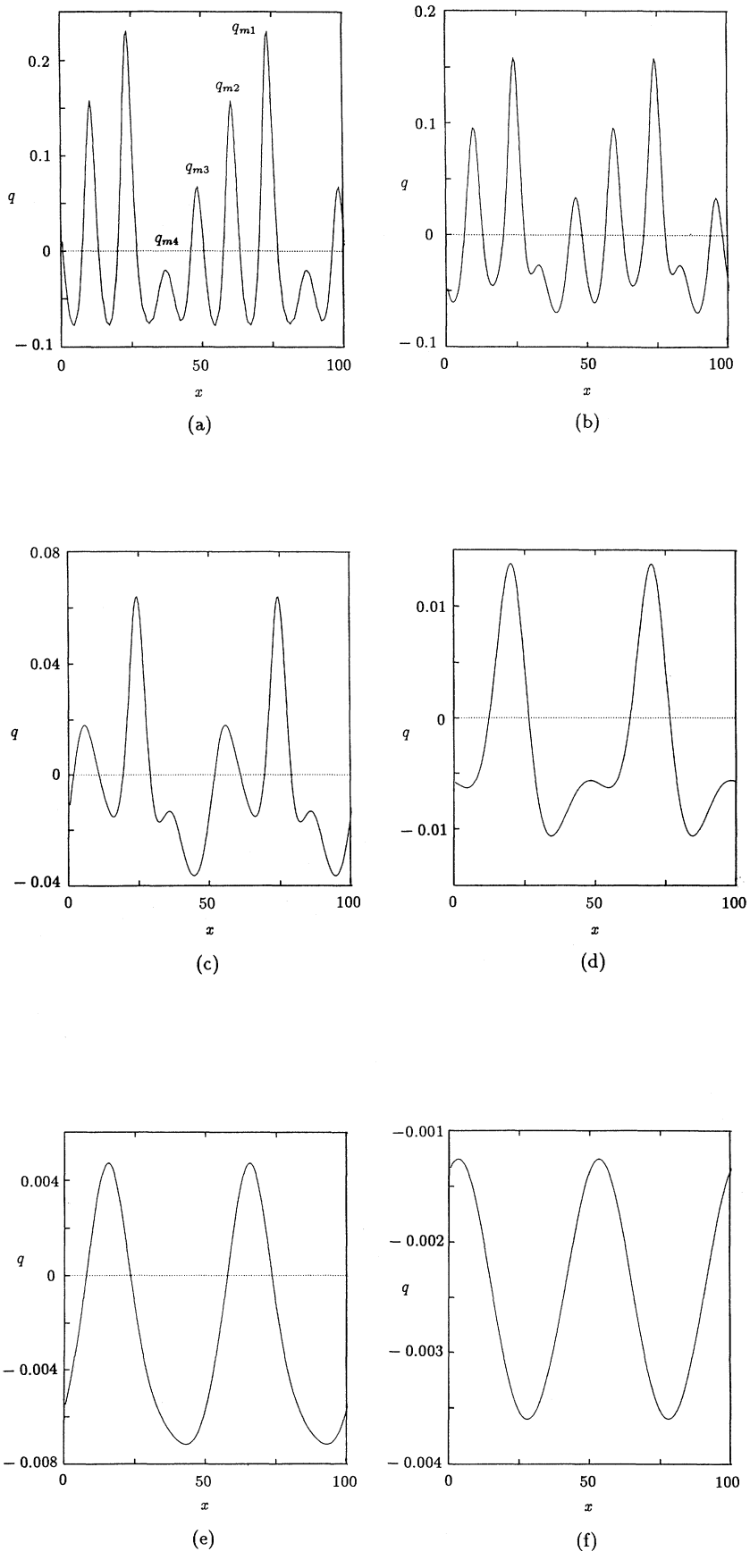

FIG. 1. Typical time evolution of soliton wave forms during self-organization: (a) initial profile at $t = 0$, with the four solitons denoted as q_{m1}, q_{m2}, q_{m3}, and q_{m4} in order of size, (b) at $t = 40$, (c) at $t = 220$, (d) at $t = 900$, (e) at $t = 1500$, and (f) at $t = 4000$.

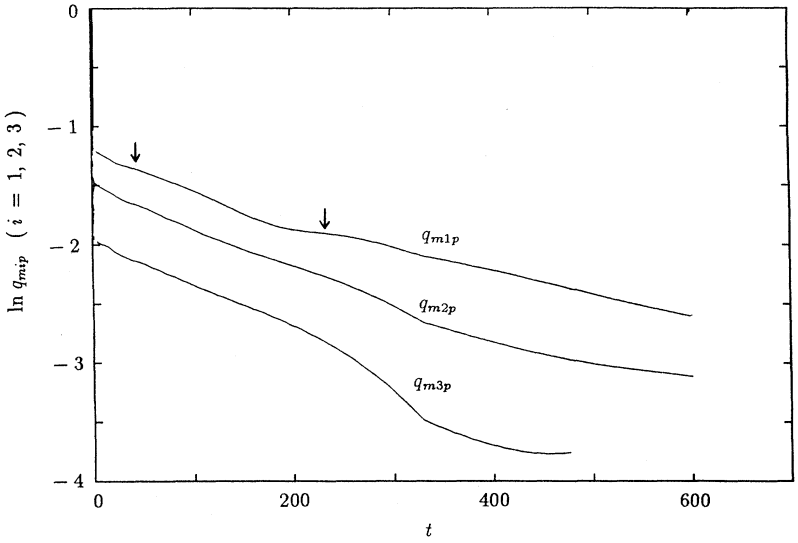

FIG. 2. Time dependence of the peak amplitudes q_{mip} (natural logarithmic scale) of the three solitons q_{mi}, with $i = 1, 2, 3$, where q_{mip} is defined approximately as the difference between the peak value of q_{mi} and the minimum value of q. The two arrows mark the two interaction phases of q_{m4} and q_{m3}, respectively, with q_{m1}.

of smaller solitons has been transferred into that of the largest one during interactions involving viscous dissipation. Concurrently, the viscous dissipation suppresses the amplitude of the first soliton q_{m1} and results in a broadening of the width of q_{m1} during the time evolution of this dynamical system. In Fig. 1(f) at $t = 4000$, we find that the lowest eigensolution of Eq. (7) has become the final self-organized state in this nonlinear dissipative system. The numerical data for Fig. 1(f) show that $q\partial q/\partial x \simeq 0.29 \times 10^{-6} < \eta\partial^2 q/\partial x^2 \simeq 0.59 \times 10^{-6}$ at $x = 22.5$. This result clearly indicates that an interchange between the dominant operators has occurred in going to the final self-similar coherent solution of Eq. (7), which is determined uniquely by the dissipative operator $\eta\partial^2 q/\partial x^2$.

Figure 2 shows the time dependence of the peak amplitudes q_{mip} (on a natural logarithmic scale) of the three solitons q_{mi}, with $i = 1, 2, 3$, where the peak amplitude q_{mip} is defined approximately as the numerical difference between the peak value of q_{mi} and the minimum value of q. The first and second arrows on the curve for q_{m1p} in Fig. 2 indicate, respectively, the two interaction phases of q_{m4} and q_{m3} with q_{m1}. From Fig. 2 it can be seen that around the time corresponding to the second arrow, from $t \sim 200$ to $t \sim 350$, during the interaction between q_{m3} and q_{m1} the decay of q_{m3p} is accelerated and the associated solition vanishes, while that of q_{m1p} is decelerated. In other words, the energy of the smaller soliton is transfered into the larger one q_{m1} during a dissipative nonlinear interaction.

Figure 3 shows the time dependence of the peak amplitude q_{m1p} (on a natural logarithmic scale) of the first soliton q_{m1} presented on a longer time scale than that of Fig. 2. After a rapid decay lasting until around $t \sim 1200$, the decay rate of q_{m1p} is seen to become almost constant. At $t = 4000$, corresponding to that in Fig. 1(f), the decay constant has a value of 0.634×10^{-3}, which agrees very well with the theoretical decay constant of $\alpha/2 = \eta\lambda_1^2 = 0.632\times10^{-3}$.

When we decompose the profile of q at each time by using the orthogonal eigenfunctions $a_k = \sin\lambda_k x$, with $\lambda_k \equiv 2\pi k/b$, which belong to the dissipative operator $\eta\partial^2 q/\partial x^2$ and arise from the associated eigenvalue problem [10] $\delta^2 F = 0$, we find that the dissipative nonlinear interaction of the solitons and the resultant broadening of the soliton width due to the viscous dissipation yield energy transfer toward both the higher and the lower spectral eigenvalues λ_k for the components a_k. At the same time, it can be seen that the dissipative operator $\eta\partial^2 q/\partial x^2$ causes the higher spectral components to dissipate more rapidly, with decay constants of $\alpha_k/2$ $(= \eta\lambda_k^2)$, while the lowest eigenmode of Eq. (7) persists to the end, as shown in Fig. 1(f).

In conclusion, the theoretical analysis and numerical simulation presented here for the self-organization of solitons described by the KdV equation with a viscous dissipation term indicate that during dissipative nonlinear interactions, two basic processes for the self-organization of solitons are energy transfer and selective dissipation among the eigenmodes of the dissipative operator $\eta\partial^2 q/\partial x^2$. Also, the numerical simulations presented in Figs. 1 and 3 show that an important process during

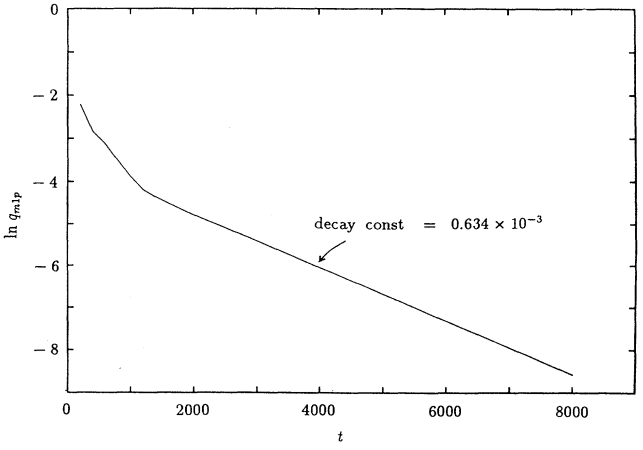

FIG. 3. Time dependence of the peak amplitude q_{m1p} (natural logarithmic scale) of the first soliton q_{m1} over a longer time scale than that used in Fig. 2.

nonlinear self-organization is an interchange between the dominant operators. This interchange had hitherto been overlooked in conventional self-organization theories. It leads to a final self-similar coherent structure that is determined uniquely by the dissipative operator.

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